Phase Transitions, 1993, Vol. 44, pp. 37-46
Reprints available directly from the publisher
Photocopying permitted by license only

BINARY TILINGS AND QUASI-QUASICRYSTALLINE TILINGS

FRÉDÉRIC LANÇON and LUC BILLARD

Département de Recherche Fondamentale sur la Matière Condensée, SP2M/MP, CENG, 85X, 38041 Grenoble cedex, France

We introduce tilings which obey vertex matching rules and generalize the five-fold binary tilings to any odd symmetry. In particular, quasicrystalline, crystalline or random binary tilings do exist. We also suggest another structure, which we call a quasi-quasicrystal and which can be obtained by iteration of a substitution.

KEY WORDS: Tiling; quasicrystal; quasi-quasicrystal; binary tiling; matching rules; substitution.

1 INTRODUCTION

The Penrose tilings (Penrose, 1974) are prototypes of two-dimensional five-fold quasicrystals. They are made of two rhombs: the large tile, L, has corner angles equal to 2 and 3 in unit of $\pi/5$ and the sharp tile, S, has corner angles equal to 1 and 4 in the same unit. We introduce other tilings made with the same tiles but such that, at any vertex, the corner angles are either all odd or all even; we call them binary tilings (Lançon and Billard, 1988). Quasicrystalline binary tilings with five-fold symmetry can be obtained from Penrose tilings by the decoration of the tiles shown in Figure 1. In fact this decoration can be applied to any tiling made of L and S tiles to produce a binary tiling.

We will show how this decoration can be generalized to any odd symmetry in order to produce $(2n + 1)$-fold binary tilings. We will also introduce the structures obtained by iteration of this decoration and we will show that they are not quasiperiodic. We call them quasi-quasicrystals. We will give some properties of these tilings using the hyper-space description and show one-dimensional analogues.

2 BINARY TILINGS

The five-fold Penrose tilings and the generalized quasicrystalline tilings of the plane with N-fold symmetry are made with tiles which are the rhombs built with the vectors $\{\mathbf{e}_i\}_{i=0,N-1}$, where $\mathbf{e}_i$ has the coordinates $(\cos(2\pi i/N), \sin(2\pi i/N))$. If N is odd, say $N = 2n + 1$, there are n tiles. Let us call T_i $(i = 1, n)$ the rhombs with edges parallel to $\mathbf{e}_0$ or $\mathbf{e}_i$ (or those obtained from these by $2\pi/N$ rotations).

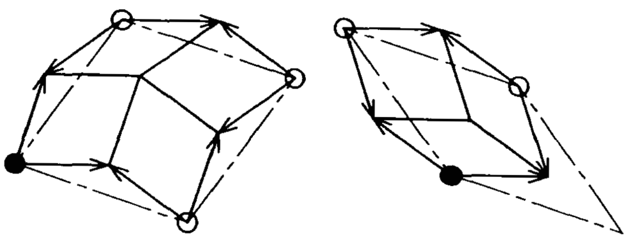

Figure 1 Decoration of the Penrose rhombs (dashed lines) with smaller ones (solid lines). Circles and arrows represent the origins and the vectors which characterize the tiles. The black disk is set on the initial rhomb origins.

To produce a binary tiling, we set new tiles on the tiles T_i. These new tiles have the same shapes as the tiles T_i but are smaller. Their edge length is divided by a factor $\theta = 2\sin(n\pi/N)$ and their orientation is rotated by $\pi/2N$, as shown in Figures 1 and 2 for $N = 5$ and 7 respectively. Let $\mathbf{f}_i$ be the N vectors which describe the edges of the new tiles. For a given initial tile, each edge $\mathbf{e}_i$ leads to two edges $\mathbf{f}_i$ and $\mathbf{f}_{i+n}$ drawn from the extremities of $\mathbf{e}_i$ and meeting at a new vertex. Thus at each vertex of the initial rhombs is set one new tile built with the two new edges starting from this vertex. Therefore each original tile leads to four new tiles except the last tile T_n for which the superposition of two new edges occurs and which leads to three new tiles. Any two adjacent tiles have their common edge which transforms itself in the same way, and this allows a perfect match between the new tiles. With a suitable renumbering of the tile types ($[n, 1, n-1, 2, \ldots] \to [1, 2, 3, 4, \ldots]$) the $n \times n$ matrix which gives the numbers of new tiles from the numbers of old tiles is

$$M = \begin{pmatrix} 2 & 1 & & & & & \\ 1 & 2 & 1 & & & & \\ & 1 & \cdot & \cdot & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & 2 & 1 & \\ & & & & 1 & 2 & 1 \\ & & & & & 1 & 3 \end{pmatrix} \quad (1)$$

(by convention the null terms are omitted in the display of the matrices).

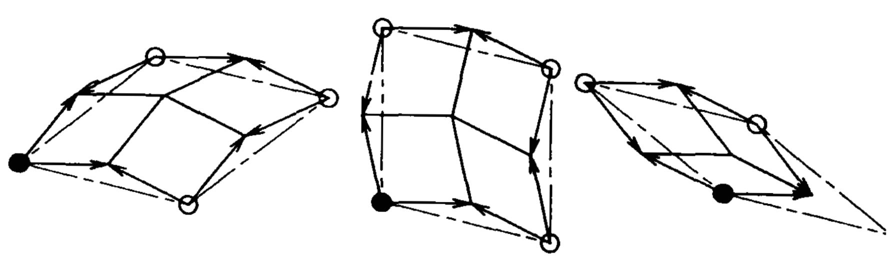

Figure 2 Decoration of the three rhombs of seven-fold tilings (dashed lines) with smaller ones (solid lines). Circles and arrows represent the origins and the vectors which characterize the tiles. The black disk is set on the initial rhomb origins.

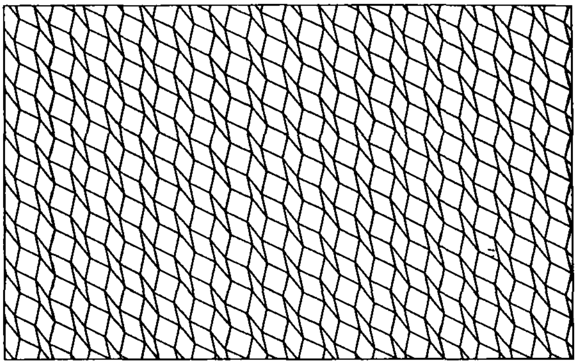

Figure 3 Periodic binary tiling.

Choosing one arbitrary vertex of the initial tiling as the origin, we associate N integer indices $(m_0, \ldots, m_{N-1})$ to any vertex of the new tiling by forming a path along the tile edges, from the origin to this vertex. Thus its position is equal to $\mathbf{r} = \sum_{i=0}^{N-1} m_i \mathbf{f}_i$. It can be easily shown that the integer sum $m_\Delta = \sum_{i=0}^{N-1} m_i$ can take only three values. It is equal to zero for all the vertices which are common to the initial and the new tiling, and one or two for the others. At vertices where m_Δ is even, all the corner angles of the rhombs are even multiples of π/N; when $m_\Delta = 1$, they are odd multiples. Thus this parity property is not limited to tilings with five-fold directions; we also call binary tilings, the tilings with N-fold directions which have this property.

Any tiling made with the tiles T_i can be decorated and thus different structures may be generated. For instance, Figures 3, 4 and 5 show respectively crystalline, quasicrystalline and random binary tilings in the case of 7-fold directions. In the next section we also present binary tilings with a less classical structure. Note that not all the binary tilings are obtained with the decoration described here.

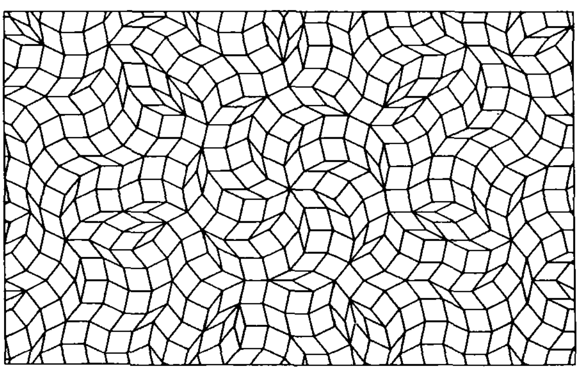

Figure 4 Seven-fold quasicrystalline binary tiling.

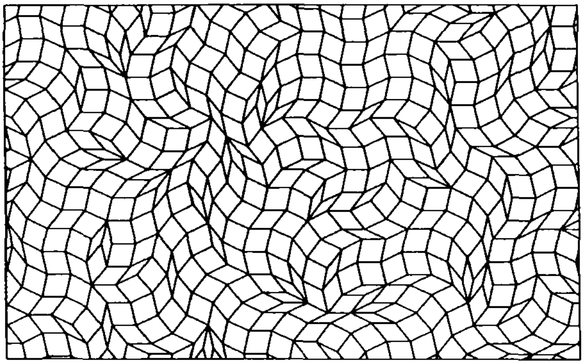

Figure 5 Seven-fold random binary tiling.

3 QUASI-QUASICRYSTALLINE TILINGS

The decoration described above has an interesting property: it can be iterated. From an initial tiling with $(2n + 1)$-fold directions, the decoration generates new tiles which can be substituted for initial ones. Up to a scaling factor, the new tiles have the same shapes as the previous ones and thus the substitution may be done again. This substitution has been described in the five-fold case (Godrèche and Lançon, 1992) and we present here the general case.

In order to keep constant the size of the tiles after each substitution, the lengths are multiplied by the factor θ. To keep the same edge orientations the tiling is also rotated by an angle $-\pi/2N$. To study the properties of this substitution we introduce the sequences of finite tilings obtained from the elementary tiles and derive a recursion formula (Aubry, Godrèche and Luck, 1988; Godrèche and Luck, 1989; Godrèche, 1989; Godrèche and Luck, 1990). Let us call $T_i^{(p)}$ the finite tiling obtained after p substitutions starting from the tiling $T_i^{(0)}$ made of only one tile T_i with vertices at O, $O + \mathbf{e}_0$, $O + \mathbf{e}_0 + \mathbf{e}_i$.

At the iteration $p + 1$ the tilings $(T_i^{(p)})_{i=1,n}$ are duplicated and joined with suitable rotations and translations to get the new tilings

$$\begin{aligned} T_i^{(p+1)} = & \quad T_i^{(p)} \\ \cup & \quad [r^i, g^{p+1}(\mathbf{e}_0)]T_{n-i}^{(p)} \\ \cup & \quad [r^{n+i}, g^{p+1}(\mathbf{e}_i)]T_{n+1-i}^{(p)} \qquad (\text{for } i \neq n) \\ \cup & \quad [r^n, g^{p+1}(\mathbf{e}_o + \mathbf{e}_i)]T_i^{(p)} \end{aligned} \tag{2}$$

where r is the rotation of angle $2\pi/N$ around the origin O; g is the similarity made of the rotation $-\pi/2N$ around O and the dilatation of factor $\theta = 2\sin(n\pi/N)$. The notation $[\mathrm{r}, \mathbf{t}]$ denotes the application to a tiling of the rotation r followed by the translation $\mathbf{t}$. The relation which gives the tiling $T_n^{(p+1)}$ is the relation (2) with $i = n$ except that the second term in the union operation is omitted. Figure 6 shows part of a 7-fold tiling obtained after successive substitutions.

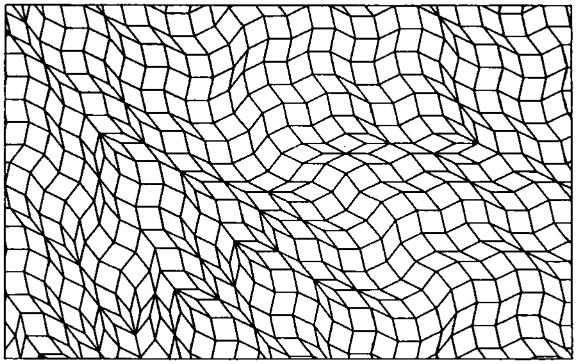

Figure 6 Seven-fold quasi-quasicrystalline binary tiling.

The application g is the application which, in Figures 1 and 2, transforms the small rhombs set on the origin (solid lines) into the large rhombs (dashed lines). More generally there is the following inflation property: g transforms any vertex of $T_i^{(p)}$ into a vertex of $T_i^{(p+1)}$ (for $p > 0$). The corresponding matrix which transforms the N integer indices of the vertices is

$$M_G = \mathbb{1} - \begin{pmatrix} 0 & & & & & 1 \\ 1 & 0 & & & & \\ & 1 & \cdot & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & 1 & 0 \\ & & & & & 1 \quad 0 \end{pmatrix}^n \tag{3}$$

where $\mathbb{1}$ is the $N \times N$ identity matrix. The N integer indices of the tiling vertices are also the coordinates in a N-dimensional space of some vertices of a hyper-cubic lattice. These points are the vertices of 2D-facets which form a rough 2D-surface into the ND-space. The tiling is the projection of this rough 2D-surface onto a plane $E^{\parallel}$, where $E^{\parallel}$ is such that the basis of the hyper-lattice projects as $(\mathbf{e}_i)_{i=0,N-1}$. On the other hand, the projection of this 2D-surface onto a perpendicular $(N-2)$D-space gives some insight into the tiling structure. For instance in the case of a Penrose tiling, the projection of the rough 2D-surface is contained in a finite domain which is the projection of the ND-cubic cell. This implies that the Fourier Transform of the Penrose tiling is made of Bragg peaks (Duneau and Katz, 1985; Elser, 1986; Katz and Duneau, 1986).

Let us now show that, for the tilings obtained with the substitution (2), the rough 2D-surfaces project into domains whose sizes diverge when the number of iterations increases. Any vertex of these tilings is transformed by g^p into another vertex after p iterations and the coordinates in the ND-space are transformed by G^p, where G is represented in the basis of the cubic lattice by M_G. We can choose a basis of the

ND-space in which G is represented by the matrix

$$M'_G = \begin{pmatrix} \gamma_1 & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \cdot & & \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \gamma_2 & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ & & \cdot & \gamma_n & \begin{matrix} 0 \\ 0 \end{matrix} \\ & & & \begin{matrix} 0 & 0 \end{matrix} & 0 \end{pmatrix} \tag{4}$$

The last diagonal term of the matrix M'_G is zero. It corresponds to the 1D globally invariant space Δ which contains the diagonal of the hyper-cube. Thus the sum m_Δ of the indices is null for all the vertices which are obtained by the inflation of the vertices at the previous substitution.

The 2×2 block γ_1 corresponds to $E^{\parallel}$ and represents the similarity g: as said above the growing factor of the tiling is θ, the scaling factor of g.

More generally each 2×2 block γ_i corresponds to a globally invariant plane E_i and represents a similarity of scaling factor θ_i equal to $2|\sin(n\pi i/N)|$. Thus for any odd symmetry $N(>3)$ there are always, at least, two scaling factors greater than one: θ_1 which corresponds to the inflation factor in $E^{\parallel}$ and another one which corresponds to an inflation in the perpendicular space (θ_2 for $N = 5$ and, for instance, θ_3 for $N \geq 7$). Therefore, for an infinite tiling (i.e. when the iteration $p \to \infty$), the vertices in perpendicular space are not included in a finite domain.

For a tiling with $\mathcal{N}$ vertices whose corresponding components in a 2D perpendicular space E_i are the vertices $\mathbf{x}^{\perp}$, we can compute the mean-square width of the vertex distribution in perpendicular space E_i:

$$h^2 = \frac{1}{\mathcal{N}} \sum_{j=1}^{\mathcal{N}} \left(\mathbf{x}_j^{\perp} - \overline{\mathbf{x}^{\perp}}\right)^2 \tag{5}$$

In five-fold tilings obtained after a large number of substitutions, Figure 7 shows that h^2 increases with $\mathcal{N}$ as:

$$h^2 = c\mathcal{N}^{\alpha} \tag{6}$$

where c is some constant and the exponent is $\alpha = \ln \theta_2 / \ln \theta_1 \simeq 0.2516$. This is in contrast with the quasicrystalline tilings where h^2 tends to a constant or with the random tilings where the phason fluctuation $\langle h^2 \rangle$ grows logarithmically (Widom, Deng and Henley, 1989; Strandburg, Tang and Jaric, 1989; Strandburg, 1989). The sequences of tilings obtained by the substitution described here tend to tilings that we can call quasi-quasicrystals.

4 1D QUASI-QUASICRYSTALS

The Fibonacci sequence is a 1D prototype of quasicrystals and is analogous to the Penrose tiling. We introduce 1D sequences which, in the same way, are analogous to the quasi-quasicrystalline tilings described above.

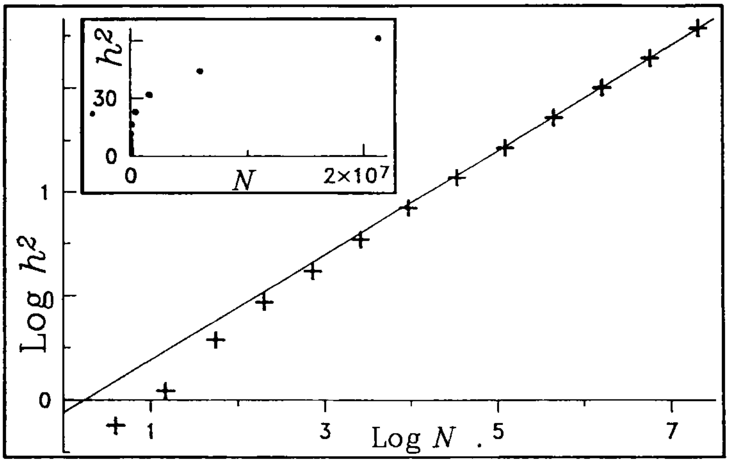

Figure 7 For the five-fold quasi-quasicrystalline tilings $T_1^{(p)}$, mean-square width, h^2, of the vertex distribution in perpendicular space E_2 versus the number, N, of tiles at iterations $p = 1$ to 13. The thin line is the straight-line of slope $\alpha = \ln \theta_2 / \ln \theta_1$ which is drawn through the last point (corresponding to $N = 21\,312\,500$ tiles).

The 1D sequences analogous to the $(2n + 1)$-fold tilings are made with a set of n segments $\{T_i\}_{i=1,n}$. We define the following substitution rule:

$$\left| \begin{array}{ll} T_i \to T_i T_{n-i} T_{n+1-i} T_i & (\text{for } i < n) \\ T_n \to T_n T_1 T_n & \end{array} \right. \tag{7}$$

The matrix M (Eq. 1) gives the relation between the number of segments before and after one substitution (with the renumbering: $[n, 1, n-1, 2, \ldots] \to [1, 2, 3, 4, \ldots]$). To each vertex of a sequence is associated a set of n integers m_i which defines its position $x = \sum_{i=1}^{n} m_i e_i$, where e_i is the length of the segment T_i. These indices are the coordinates of a vertex of a nD hypercubic lattice. Thus any sequence corresponds, in the nD space, to a staircase curve made of segments parallel to the hypercubic axes. The matrix M also transforms the indices of a vertex which belongs to a sequence into the indices of a vertex which belongs to the new sequence obtained after one substitution. The eigenvalues of matrix M are $\Theta_i = \theta_i^2 = 4\sin^2[n\pi i/(2n+1)]$ and let us call E_i the corresponding 1D eigenspaces.

Θ_1 is the largest eigenvalue and E_1 can be identified with the tiling space. The lengths e_i of the segments are the lengths of the projections onto E_1 of the hypercubic lattice basis and are proportional to $\sin[2\pi i/(2n+1)]$. The scaling factor of the sequence is $\Theta = \Theta_1 = \theta^2$, i.e., the square of the 2D tiling scaling factor.

On the other hand, after each substitution, the initial and final projections of the staircase curves onto any eigenspace scale with the corresponding eigenvalue. Since, besides Θ_1, at least one of the other eigenvalues is larger than unity, the projection of the staircase curve onto the corresponding eigenspace covers a domain with an increasing size. Therefore the staircase curve is not included in a strip parallel to E_1 with a bounded width.

Examples of such non quasicrystalline 1D structures obtained by substitution rules have already been given. Bombieri and Taylor (1986, 1987), have discussed the existence of Bragg peaks in the diffraction spectrum depending on the number of eigenvalues of the substitution matrix which have a modulus larger than unity. Godrèche and Luck (1990) have conjectured that, generically, when this number is larger than one the diffraction spectrum is singular continuous.

We now discuss in more details the case $n = 2$. Let us call L the segment T_1 and S the segment T_2. With these notations the substitution rule (7) becomes:

$$\left| \begin{array}{l} L \to LLSL \\ S \to SLS \end{array} \right. \tag{8}$$

Figure 8 shows in the 2D hyper-space the staircase curve which corresponds to the sequence obtained by iteration of this substitution rule starting from one L segment.

The Fibonacci sequence can be obtained with the substitution rule:

$$\left| \begin{array}{l} L \to LS \\ S \to L \end{array} \right. \tag{9}$$

The matrix M relates the numbers $\mathcal{N}_L$ and $\mathcal{N}_S$ of L and S segments before and after one substitution (8):

$$\begin{pmatrix} \mathcal{N}_S \\ \mathcal{N}_L \end{pmatrix}^{(\mathrm{new})} = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} \mathcal{N}_S \\ \mathcal{N}_L \end{pmatrix}^{(\mathrm{old})} \tag{10}$$

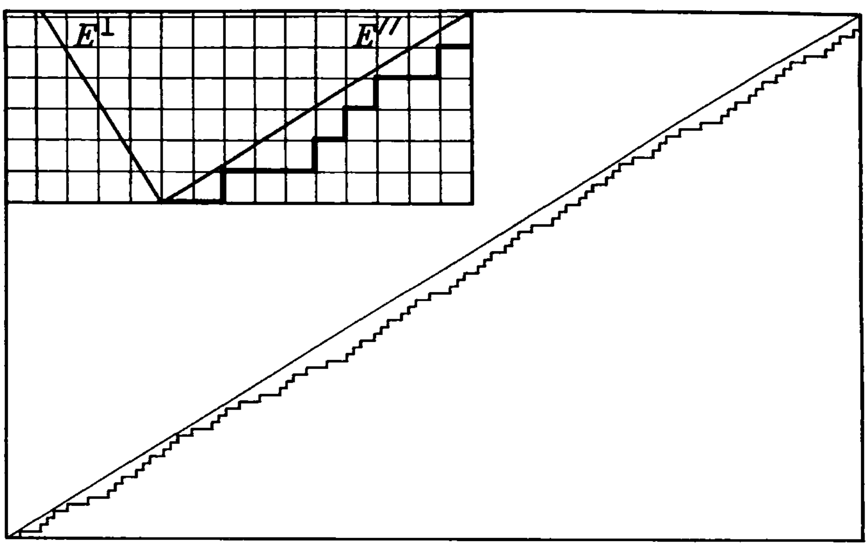

Figure 8 Representation in the plane of a quasi-quasicrystalline sequence made of two segments which are the projections, onto the physical space $E^{\parallel}$, of the horizontal and vertical unit square edges.

while for the Fibonacci sequence we have the relation:

$$\begin{pmatrix} \mathcal{N}_S \\ \mathcal{N}_L \end{pmatrix}^{(\text{new})} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \mathcal{N}_S \\ \mathcal{N}_L \end{pmatrix}^{(\text{old})} \tag{11}$$

Both substitution matrices have the same eigenspaces, the lines $E_1 = E^{\parallel}$ and $E_2 = E^{\perp}$. Therefore the lengths of the segments L and S are the same in both cases, i.e., respectively proportional to τ and 1, where τ is the golden ratio $2\cos(\pi/5) = (1+\sqrt{5})/2$. The ratio $\mathcal{N}_L/\mathcal{N}_S$ also tends to the same limit τ in both cases when the number of substitutions tends to infinity.

However the eigenvalues are equal to $\Theta_1 = \tau + 2 \simeq 3.618$ and $\Theta_2 = 3 - \tau \simeq 1.382$ in the quasi-quasicrystal case, while they are equal to $\tau \simeq 1.618$ and $1 - \tau \simeq -0.618$ in the Fibonacci case. The projection of the staircase curve onto the perpendicular space $E^{\perp}$, which is characterized by $|\Theta_2|$, is not contained in a bounded domain in the quasi-quasicrystal case since $|\Theta_2| > 1$; in the Fibonacci case ($|\Theta_2| < 1$), this projection is contained in a segment which is the projection of a unit cell of the square lattice onto $E^{\perp}$.

To describe more precisely the perpendicular space behavior of the quasi-quasicrystalline sequence, let us introduce the density $\rho_p(x^{\perp})$ of the vertices $x^{\perp}$ in $E^{\perp}$ which are the projections of the vertices of the staircase curve after p substitutions starting from one initial segment. One can show that the moment of order r of the distribution $\rho_p(x^{\perp})/\mathcal{N}_p$ varies like Θ_2^{rp} when $p \to \infty$. In particular the mean-square

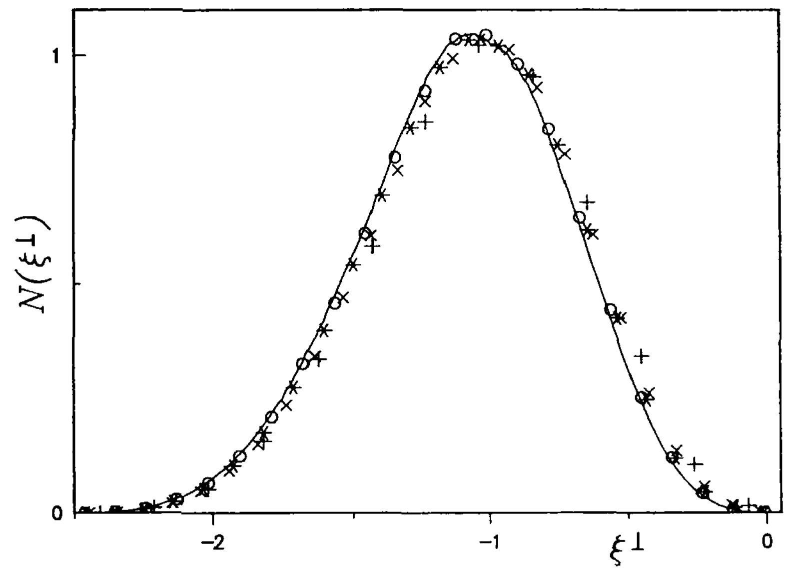

Figure 9 Distributions $N(\xi^{\perp})$ of the variable $\xi^{\perp} = x^{\perp}/\Theta_2^p$. The value of $x^{\perp}$ are the perpendicular coordinates corresponding to the vertices of the quasi-quasicrystalline sequence of L and S segments; the sequences are obtained by applying the substitution $[L \to LLSL,\ S \to SLS]$ to one initial L segment p times; $\Theta_2 = 3 - \tau$ is the eigenvalue corresponding to the eigenspace $E^{\perp}$. The distributions are represented at iterations p equal to 6 (+), 7 (×), 8 (∗), 9 (∘) and 10 (continuous curve).

width h^2 increases like $\mathcal{N}^\alpha$, where $\alpha = 2 \ln \Theta_2/\ln \Theta_1 \simeq 0.5031$. Figure 9 shows that the distribution of the reduced variable $\xi^\perp = x^\perp/\Theta_2^\xi$ tends to a limit distribution $\Gamma(\xi^\perp)$.

5 CONCLUSION

In this paper we have shown that binary tilings can be generalized to any odd symmetry. In these tilings the vertices fall into two classes according to corner angles values and also fall into three classes according to the values of their indices sum. These properties allow the decoration of the tiles by particles or spins on the vertices.

A special class of tilings obtained by iteration of a substitution rule has been introduced. While they are made with the same tiles as the quasiperiodic tilings, we have shown that they are not quasiperiodic. They correspond to unbounded covering of some perpendicular spaces and we have called them quasi-quasicrystals.

We have studied one dimensional sequences closely related to these two dimensional tilings. In particular, using the two segments of the Fibonacci sequences we can build quasi-quasicrystalline sequences with the same frequencies of the segments as in the Fibonacci case. The distribution of perpendicular coordinates, properly reduced by a factor which exponentially increases with the iteration number, approaches a limit distribution.

References

Aubry S., C. Godrèche and J. M. Luck. (1988). Scaling properties of a structure intermediate between quasiperiodic and random. *J. Stat. Phys.*, **51**, 1033–1075.

Bombieri E. and J. E. Taylor. (1986). Which distributions of matter diffract? An initial investigation. *J. Phys. France Coll.*, *C* **3**, 19–28.

Bombieri E. and J. E. Taylor. (1987). Quasicrystals, tilings, and algebraic number theory; some preliminary connections. *Contemp. Math.*, **64**, 241–264.

Duneau M. and A. Katz. (1985). Quasiperiodic patterns. *Phys. Rev. Lett.*, **54**, 2688–2691.

Elser V. (1986). The diffraction pattern of projected structures. *Acta. Cryst.*, *A* **42**, 36–43.

Godrèche C. (1989). The sphinx: a limit-periodic tiling of the plane. *J. Phys.*, *A* **22**, L1163–L1166.

Godrèche C. and F. Lançon. (1992). A simple example of a non-Pisot tiling with five-fold symmetry. *J. Phys. I France*, **2**, 207–220.

Godrèche C. and J. M. Luck. (1989). Quasiperiodicity and randomness in tilings of the plane. *J. Stat. Phys.*, **55**, 1–28.

Godrèche C. and J. M. Luck. (1990). Multifractal analysis in reciprocal space and the nature of the Fourier transform of self-similar structures. *J. Phys.*, *A* **23**, 3769–3797.

Katz A. and M. Duneau. (1986). Quasiperiodic patterns and icosahedral symmetry. *J. Physique*, **47**, 181–196.

Lançon F. and L. Billard. (1988). Two-dimensional system with a quasi-crystalline ground state. *J. Phys. France*, **49**, 249–256.

Penrose R. (1974). The role of aesthetics in pure and applied mathematical research. *Bull. Inst. Math. and its Appl.*, **10**, 266–271.

Strandburg K. J. (1989). Random-tiling quasicrystal. *Phys. Rev.*, *B* **40**, 6071–6084.

Strandburg K. J., L. -H. Tang and M. V. Jaric. (1989). Phason elasticity in entropic quasicrystals. *Phys. Rev. Lett.*, **63**, 314–317.

Widom M., D. P. Deng and C. L. Henley. (1989). Transfer-matrix analysis of a two-dimensional quasicrystal. *Phys. Rev. Lett.*, **63**, 310–313.